Def. An ODE is linear if $y, y', \dots, y^{(n)}$ appear linearly in the eqn. (to the first power, NOT inside other functions)

The general linear ODE of order n can be written in the form:

$$\boxed{a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0(x)\, y = f(x)}$$

Special case if $\boxed{f(x) \equiv 0}$ then the eqn. is linear homogeneous

Very Special case: if $a_n(x), a_{n-1}(x), \dots, a_0(x)$ are constant functions then linear ODE with constant coefficients.

ex. $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$ (constant coeff & homogen)

e.g. $y'' - 5y' + 6y = 0$ In this case we can associate a polynomial.

associate: $t^2 - 5t + 6 = 0$ $t_1 = 2$; $t_2 = 3$

solution: $C_1 e^{2x} + C_2 e^{3x}$

Fundamental Theorem of Algebra

The polynomial equation

$$a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0 = 0$$

has exactly n solutions, PROVIDED we count multiplicities and admit complex solutions (even for real coefficients).

Analogous to the F.T. of A. is the F.T. of Arithmetic: Given $\mathbb{N} = \{1, 2, 3, \dots\}$ integers, $\mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$ primes, any $n \in \mathbb{N}$ is

$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_n^{\alpha_n}$ (has an UNIQUE FACTORISATION)

Fundamental Th. of Algebra (2^{nd} form)

$a_n t^n + \dots\dots + a_1 t + a_0 = a_n (t - t_1)(t - t_2) \dots (t - t_n)$

where $t_1, t_2, \dots, t_n \in \mathbb{C}$ and some of them could be repeated.

For polynomials: ex. $p(x) = (x+1)(x-2)^2 = (x+1)(x^2-4x+4) =$
$= x^3 - 4x^2 + 4x = x^3 - 3x^2 + 4$

3 factors, one of which is repeated twice
(has multiplicity = 2)

For polynomials of degree 1, 2 there are explicit formulas for the solution.

There are also for degree 3, 4 (formulae with $+, -, \cdot, :$ and $\sqrt{\ }$)
(Tartaglia, Bombelli, Cardano)

From degree 5 up the functions connecting coeff. & solutions are more complicated (modular functions)

There are numerical methods (Newton's method ...)

Prop. If a polynomial of degree n has real coefficients $a_n, a_{n-1}, \cdots, a_0$ and complex solutions these must appear in conjugate pairs.

This, together with Euler's Formula, gives us some typical solutions of linear ODE with constant coeff.

ex. $y'' + y = 0 \quad t^2 + 1 = 0 \longrightarrow = (t+i)(t-i)$

$$y_{\text{GEN}}(x) = c_1 e^{ix} + c_2 e^{-ix} =$$
$$= c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x) =$$
$$= \underbrace{(c_1 + c_2)}_{d_1} \cos x + \underbrace{i(c_1 - c_2)}_{d_2} \sin x$$

Check: $y_{\text{GEN}}(x) = d_1 \cos x + d_2 \sin x$

$y' = -d_1 \sin x + d_2 \cos x$

$y'' = -d_1 \cos x - d_2 \sin x$

$y + y'' = 0$ for any c_1 & c_2

More generally, to each pair of complex conjugate solutions $\alpha \pm i\beta$ of the pol. associated to our ODE, we associate

$$d_1 e^{\alpha x} \cos \beta x + d_2 e^{\alpha x} \sin \beta x \ldots$$

<u>Moral</u> Provided we solve (maybe numerically) the characteristic equation (alg. eq. associated), the solution of the ODE is a linear combination of products of exponentials, sines, cosines.

Typical term in the sum: $e^{-2x} \cos 3x$

If the coeff. of a linear homogeneous ODE are not constant, then it's no longer useful to associate a polynomial BUT the general solution still has the structure

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

where the functions $y_1(x) \cdots y_n(x)$ are <u>linearly independent</u>: i.e., for $x \in I \subseteq \mathbb{R}$, the sum

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0 \quad \text{for } \forall x \in I$$

$\Longleftrightarrow$ $c_1 = c_2 = \ldots = c_n = 0$ (it's true only for the trivial sol.)

How do we find these independent functions?

Several methods, e.g. power series.

For <u>non-homogeneous linear ODE</u> the structure of the solution is:

$$y_{GEN}(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)}_{\text{solution of the associated homogeneous eq.}} + \underbrace{\eta(x)}_{\text{forced term}}$$

Example: $y'' - 5y' + 6y = 3x + 1$

$$y(x) = y_{HOM.}(x) + \eta(x) = c_1 e^{2x} + c_2 e^{3x} + \eta(x)$$

Since $f(x)$ is a polynomial of degree 1, we can guess that also $\eta(x) = ax + b$ is a suitable polyn. of degree 1.

$\eta = ax + b \quad \eta' = a \quad \eta'' = 0$

$0 - 5a + 6(ax+b) = 3x + 1$

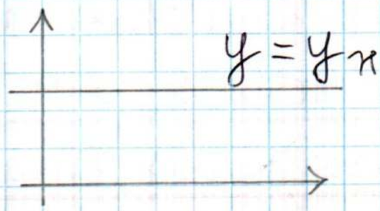
$$6ax + (6b - 5a) = 3x + 1 \Longrightarrow \begin{cases} 6a = 3 & a = 1/2 \\ 6b - 5a = 1 \end{cases}$$

$6b - 5/2 = 1 \quad 6b = \frac{7}{2} \quad b = \frac{7}{12}$

$$y_{GEN}(x) = \underbrace{c_1 e^{2x} + c_2 e^{3x}}_{\text{arbitrary constants}} + \underbrace{\tfrac{1}{2}x + 7/12}_{\text{PRECISE constants}}$$

$\boxed{\text{ODE: } y' = f(x)\, g(y)}$ SEPARATION of VARIABLES

Rem. if $y_r \in \mathbb{R}$ such that $g(y_r) = 0$, then the horizontal lines $y = y_r$ are solutions

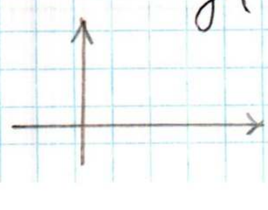


If $g(y) \neq 0$, we write $dy/dx = f(x)\, g(y)$

$\boxed{\int \frac{dy}{g(y)} = \int f(x)\, dx}$ if we can solve the $\int$ get = with 1 arb. const.

ex.1) $y' = -xy \qquad f(x) = -x \quad g(y) = y$

Trivial solution $y = 0$



$$\int \frac{dy}{y} = \int -x\,dx$$

$$\log|y| = -\frac{x^2}{2} + c$$

$$|y| = e^{-\frac{x^2}{2}+c} = \underbrace{e^c}_{k>0} \cdot e^{-x^2/2}$$

$$y(x) = c e^{-x^2/2}$$

<u>Check:</u> $y = ce^{-x^2/2}$ $\quad y' = c\left(-\frac{2x}{2}\right)e^{-x^2/2}$

$$y' = -xy \;\checkmark$$

<u>Ex. 2)</u> $$y' = \frac{y(x^3-2x-3)}{y^2+1}$$

$y=0$ trivial sol. $\quad g(y) = \frac{y}{y^2+1} = 0$

$$\int \frac{y^2+1}{y}\,dy = \int (x^3+2x-3)\,dx$$

This is a trascendent function (can't explicit the y)

$$\frac{y^2}{2} + \log|y| = \frac{x^4}{4} + x^2 - 3x + c \qquad (*)$$

this defines all the solutions to our ODE <u>implicitly</u>.

<u>Obs. 1)</u> numerically, using a computer, we can draw $y(x)$ from $(*)$.

<u>2)</u> We could write $y(x) = \sum_{k=0}^{\infty} c_k x^k$ and compute c_k from $(*)$ $\Rightarrow$ try to do as an exercise.

So there is a <u>serious</u> reason to talk about <u>series</u> (this is a pun!)