LESS. 6 13/11/2008

ORTHOGONAL SYSTEMS OF FUNCTIONS (chapter 2)

We start from definition of scalar product (f,g)

$$(f,g) \stackrel{(*)}{=} \int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx$$

There are several possible scalar products, depending on the choice of the interval $[a;b]$, and the weight function $w(x) \geq 0$ (a very simple choice of weight is $w(x)=1$ or $w(x)=c$ a constant)
In classical Fourier Series $[a;b]=[-\pi;\pi]$, but it could be any interval of length $T>0$ (period of f). The interval could also be $[0;+\infty)$ a half-line or $(-\infty;+\infty)$ a whole line.
Chosen a specific scalar product

<u>Def</u>. we say that $f \perp g$ with respect to the scalar product $(*)$ if and only if $\int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx = 0$

(the integral $=0$ because of "cancellation effect" of positive and negative areas)

<u>Remark</u> $(f,f) = \int_a^b |f(x)|^2 w(x)\,dx \geq 0$

$$\rotatebox{90}{$\equiv$}\ \|f\|^2 \text{ on } L^2([a,b],w)$$

Consider a set of (∞ many) functions $\phi_n(x)$ with $x \in (a,b)$, and $n \in I$ (set of indices, usually $\mathbb{N}^+, \mathbb{N}^0, \mathbb{Z}$).
This is called **orthogonal system** if:

(1) $\phi_n \in L^2([a,b],w) \quad \forall n \in I$

(2) $\|\phi_n\|_2 > 0 \quad \forall n \in I$ where
(the norm must be positive) $\|\phi_n\|_2 = \|\phi_n\|_{L^2([a,b],w)} = \left(\int_a^b |\phi_n(x)|^2 w(x)\,dx\right.$

(3) $(\phi_n, \phi_m) = 0$ if $n \neq m$
(these functions are $\perp$ to each other)

Example (important)

$$\phi_n(x) = e^{inx} = \cos nx + i \sin nx$$

$$I = \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

$[a; b] = [-\pi; \pi]$ (actually any interval of length 2π is OK, e.g. $[0; 2\pi]$ or $[-\frac{\pi}{4}; \frac{7}{4}\pi]$)

$w(x) = 1 \quad L^2([-\pi;\pi], dx)$

(1) $\int_{-\pi}^{\pi} |e^{inx}|^2 dx = \int_{-\pi}^{\pi} 1 dx = 2\pi \Rightarrow \|\phi_n\|_2 = \sqrt{2\pi}$

$|e^{inx}|^2 = |\cos nx + i \sin nx|^2 = \cos^2 nx + \sin^2 nx = 1$

(2) $\|\phi_n\|_2 = \sqrt{2\pi}$

N.B.: (1) is almost trivial in this case, because ϕ_n itself is bounded and $\int_a^b |\phi_n(x)|^2 dx = \infty$ impossible, BUT for general orthogonal systems the $\phi_n(x)$ could be unbounded; as long as $\|\phi_n\| < \infty$ FINITE.

(3) $(\phi_n, \phi_m) = \int_{-\pi}^{\pi} \underbrace{e^{inx}}_{\phi_n} \underbrace{e^{-imx}}_{\overline{\phi_m}} dx = \int_{-\pi}^{\pi} e^{i(n-m)x} dx =$ with $n-m \neq$ (INTEGER

$= \left[\frac{e^{i(n-m)x}}{i(n-m)} \right]_{-\pi}^{\pi} = 0$ REMEMBER $e^{i\pi} = e^{-i\pi}$

$\equiv 0$ because of cancellation

<u>Def.</u>: suppose $f(x) \in L^1([a,b], w)$ and also $f(x)\overline{\phi_n(x)} \in L^1([a,b], w$ for $n \in I$, then

$$\boxed{c_n = \frac{1}{\|\phi_n\|_2^2} \cdot (f, \phi_n)}$$

are the <u>generalized Fourier coefficients</u> of $f(x)$ w.r.t. the orthogonal system $\{\phi_n\}$.

N.B. There are many orthogonal systems besides the classical e^{inx} ...

N.B. When $\phi_n(x) = e^{inx}$, $I = \mathbb{Z}$, then c_n are the classical Fourier coeff's of the 2π-periodic function f:

$$c_n = \underset{\|\phi_n\|_2^2}{\frac{1}{2\pi}} \underbrace{\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx}_{(f,\, e^{inx})}$$

N.B. if instead of 2π-periodic functions we want to consider T-periodic functions, then $\phi_n(x) = e^{i\frac{2\pi}{T}nx}$, $[a,b] = \left[-\frac{T}{2}; \frac{T}{2}\right]$, or any interval of length T, like e.g. $[0,T]$.

In the T-periodic case, our scalar product will be: $(w=1)$

$$(f,g) = \int_{-T/2}^{T/2} f(x)\, \overline{g(x)}\, dx$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-i\frac{2\pi}{T}nx}\, dx$$

Remark F.S. are well adapted both to the case of T-periodic functions and the case of functions that are only defined in $[a;b]$, with $(b-a=T)$, because we can always assume that a function defined only on $[a;b]$ is periodic outside.

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Guideline (Road Map) to apply Fourier Theory to PDE's: (1) Domain has a side which is a finite interval $\Rightarrow$ Fourier Series (classical or generalized) (2) Domain has a full line in the boundary $\Rightarrow$ Fourier Transform (3) Domain has a half line on the boundary $\Rightarrow$ Laplace Transform

The expression $\boxed{\sum_{n \in I} c_n \phi_n(x)}$ is called the expansion of $f(x)$ as **Fourier Series (Generalized)** w.r.t. the orthogonal system $\{\phi_n\}$.

This F.S. converges to $f(x)$ at least in the L^2 sense

e.g. $\lim_{N\to\infty} \| f - \sum_{n=-N}^{N} c_n \phi_n(x) \|_2 = 0$

Then, if f is "good enough" (for example $f \in C^1[a;b]$), then we have other kinds of convergence ($C^1 \Rightarrow$ pointwise, uniform ...)

N.B. If $I = \mathbb{Z}$ the partial sums are from $-N$ to N (balanced, symmetric, finite sums). If $I = \{0, 1, 2, 3, \dots\}$, then the partial sums are from 0 to N, etc.

Def. An orthogonal system is called **ortho-normal** if

$$\boxed{\|\phi_n\|_2 = 1 \quad \text{for} \quad n \in I}$$

Any orthogonal system can be normalized, just multiplying each ϕ_n by a suitable constant.

Theorem (minimizing property of Fourier Coefficients)

$f(x) \sim \sum_{n\in I} c_n \phi_n(x) \qquad f \in L^2([a;b]) \quad c_n = \frac{1}{\|\phi_n\|_2} \cdot (f, \phi_n)$

$$\boxed{\| f(x) - \sum_{j=-N}^{N} c_j \phi_j(x) \|_2 \leq \| f(x) - \sum_{j=-N}^{N} d_j \phi_j(x) \|_2}$$

and the $=$ holds $\iff d_j = c_j$ (Fourier coeff's $\forall j$)

We could say that the Fourier partial sums ($\sum_{-N}^{N} \dots$) give us the best L^2 projection on a vector sub-space of L^2 of finite dimension $(2N+1)$ (Proof later...)

Corollary $\boxed{\| f(x) - S_N(x) \|_2^2 = \| f \|_2^2 - \sum_N |c_j|^2 \| \phi_j \|_2^2}$

Corollary of the corollary **(Bessel's inequality)**:

$$\boxed{\sum_{j\in I} |c_j|^2 \|\phi_j\|_2^2 \leq \|f\|_2^2}$$

because $\| f(x) - S_N(x) \|_2$ is positive, so $f(x) > S_N(x)$

Def. We say that $\{\phi_n\}$ orthogonal system has the **Parseval Property** if, for $\forall f \in L^2([a;b])$, we have the equality in the Bessel's inequality:

$$\boxed{\sum_{j\in I} |c_j|^2 \|\phi_j\|_2^2 \underset{(*)}{=} \|f\|_2^2}$$

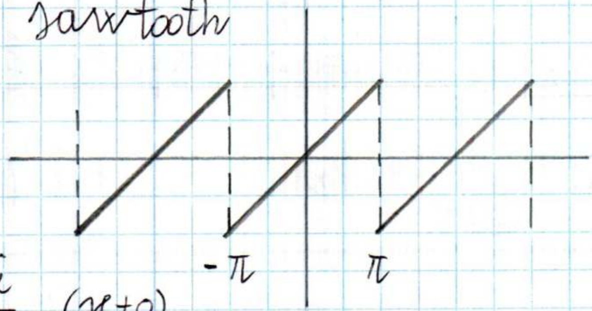
(∞-dimensional Pithagorean theorem)

Def. An orthogonal system $\{\phi_n\}$ is **complete** if no other function can be added to it, preserving properties (1) (2) (3) (in particular prop. (3), orthogonality).

> **Theorem**: if an orthogonal system has the Parseval Property (*), then it is complete.

Example:

sawtooth



$$f(x) \begin{cases} x \text{ for } x \in (-\pi;\pi) \\ 2\pi\text{-periodic} \end{cases}$$

$$f(x) = \sum_{k=-\infty}^{+\infty} c_k e^{ikx} \qquad c_k = \frac{(-1)^k i}{k} \quad (k \neq 0)$$

Th. $\phi_k(x) = e^{ikx}$ is complete in $L^2([-\pi;\pi])$ $k \in I = \mathbb{Z}$ if

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \left|\frac{(-1)^k i}{k}\right|^2 \cdot \underbrace{\|\phi_k\|_2^2}_{2\pi} = \|f\|_2^2$$

$$\downarrow \int_{-\pi}^{\pi} x^2 dx$$

$$2\pi \sum_{k\neq 0} \frac{1}{k^2} = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$2\pi \left(\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=-1}^{-\infty} \frac{1}{k^2}\right) = \frac{1}{3}(\pi^3 + \pi^3) \qquad 4\pi \overbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}^{\pi^2/6} = \frac{2}{3}\pi^3$$

$$\frac{4\pi^3}{6} = \frac{2}{3}\pi^3$$

The equality is true, so the orthogonal system is complete!

(def.) (1) (2) (3)

Complete Orthogonal Systems of functions in L^2, $\phi_\varkappa(x)$

We say that $\{\phi_\varkappa\}$, $\varkappa \in I$, is ortho-normal if

$$\|\phi_\varkappa\|_2 = 1 \qquad \forall \varkappa \in I$$

N.B. Any orthogonal system can be normalized just multipeying $\phi_\varkappa(x)$ by a constant $\frac{1}{\|\phi_\varkappa\|_2}$

Th. 2.3.1 (p. 33) (minimizing property of Fourier Coeff's)
Suppose $\{\phi_\varkappa(x)\}_{\varkappa \in I}$, $x \in (a, b)$, is an orthogonal system, $f \in L^2(a,b)$, $f \sim \sum_{\varkappa \in I} c_\varkappa \phi_\varkappa(x)$

$$\Rightarrow \left\| f(x) - \sum_{\varkappa=-n}^{n} c_\varkappa \phi_\varkappa(x) \right\|_2 \overset{(*)}{\leq} \left\| f(x) - \sum_{\varkappa=-n}^{n} d_\varkappa \phi_\varkappa(x) \right\|_2$$

and the = in (*) holds $\Leftrightarrow$ $c_\varkappa = d_\varkappa \quad \forall \varkappa$

N.B. $c_\varkappa$ is the (generalized) Fourier coefficient of $f(x)$ with respect to the orthogonal system $\{\phi_\varkappa(x)\}$ i.e.

$$c_\varkappa = \frac{1}{\|\phi_\varkappa\|_2^2} \cdot (f, \phi_\varkappa)$$

(*) means the "square deviation" of $f(x)$ from $\sum_{\varkappa=-n}^{n} d_\varkappa \phi_\varkappa(x)$ is minimal when $d_\varkappa = c_\varkappa$, where $c_\varkappa$ are the Fourier coeff. of f w. r. t. (with respect to) $\{\phi_\varkappa\}$

Proof: $\underset{\underset{f(x)}{|||}}{\| f} - \sum_{-n}^{n} d_k \underset{\underset{\phi_k(x)}{|||}}{\phi_k} \|_2^2 = \left(f - \sum_{k=-n}^{n} d_k \phi_k,\ f - \sum_{-n}^{n} d_k \phi_k\right) =$

(Scalar product)

using distributive property $= (f,f) - \left(f, \sum_{-n}^{n} d_k \phi_k\right) - \left(\sum_{-n}^{n} d_k \phi_k, f\right) +$

$+ \left(\sum_{-n}^{n} d_k \phi_k, \sum_{-n}^{n} d_k \phi_k\right) = \| f \|_2^2 - \sum_{-n}^{n} (f, d_k \phi_k) +$

$- \sum_{-n}^{n} (d_k \phi_k, f) + \sum_{j=-n}^{n} \sum_{k=-n}^{n} (d_k \phi_k, d_j \phi_j) =$

Remark
$$\boxed{\begin{aligned} (\alpha f, g) &= \alpha (f,g) \\ (f, \alpha g) &= \overline{\alpha} (f,g) \end{aligned}}$$

$= \| f \|_2^2 - \sum_{k=-n}^{n} \overline{d_k} (f, \phi_k) - \sum_{k=-n}^{n} d_k (\phi_k, f) +$

$+ \sum \sum d_k \overline{d_j} (\phi_k, \phi_j) = (*)$ $\quad (\phi_k, \phi_j) \neq 0$ when $k = j$

Remark
$$\boxed{\begin{aligned} (f, \phi_k) &= c_k \| \phi_k \|_2^2 \quad (\text{Fourier coeff. } c_k \text{ from definition}) \\ (\phi_k, f) &= \overline{(f, \phi_k)} = \overline{c_k} \| \phi_k \|_2^2 \end{aligned}}$$

$(*) = \| f \|_2^2 - \sum_{-n}^{n} \overline{d_k} c_k \| \phi_k \|_2^2 - \sum_{-n}^{n} d_k \overline{c_k} \| \phi_k \|_2^2 +$

$+ \sum_{k=-n}^{n} d_k \overline{d_k} \| \phi_k \|_2^2 = + \sum_{-n}^{m} c_k \overline{c_k} \| \phi \|_2^2 - \sum_{-n}^{m} c_k \overline{c_k} \| \phi \|_2^2$

collect these terms

$= \| f \|_2^2 - \sum_{k=-n}^{n} c_k \overline{c_k} \| \phi_k \|_2^2 + \sum_{k=-n}^{n} (c_k - d_k)(\overline{c_k} - \overline{d_k}) \| \phi_k \|_2^2 =$

$= \| f \|_2^2 - \sum_{k=-n}^{n} |c_k|^2 \| \phi_k \|_2^2 + \sum_{k=-n}^{n} |c_k - d_k|^2 \| \phi_k \|_2^2 \qquad (\star)$

This quantity is minimal $\iff$ $c_k = d_k$ for $k = -n, -n+1, \dots, n-1, n$

(QED)

Furthermore If $c_k = d_k$ $S_n f(x) = \sum_{k=-n}^{n} c_k \phi_k(x)$

Partial Sums Order n (bilateral) of our F.S. (generalized)

Then the last term in (✯) formula is $=0$ =>

$$\boxed{\|f(x) - S_n f(x)\|_2^2 = \|f\|_2^2 - \sum_{k=-n}^{n} |c_k|^2 \|\phi_k\|_2^2} \quad \left(\text{proof of corollary}\right)$$

This observation ("furthermore") implies proof of Bessel's Inequality:
$$\boxed{\|f\|_2^2 \geq \sum_{k=-n}^{n} |c_k|^2 \|\phi_k\|_2^2}$$

Def. we say that $\{\phi_k\}_{k\in I}$ has the Parseval Property if $=$ holds in
$$\boxed{\|f\|_2^2 = \sum_{k\in I} |c_k|^2 \|\phi_k\|_2^2}$$

> Theorem (no proof in this 5 credits course!)
> An orthogonal system of functions $\{\phi_k(x)\}_{k\in I}$ is complete <=> it has the Parseval Property.

Let's revisit these facts in the classical case when $\{\phi_k(x) = e^{ikx}\}$ and $I = \mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$

Also, we observed that, in the classical case, we can write down in 2 forms the F.S. of $f(x)$
$$\sim \sum_{k=-\infty}^{+\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

N.B. $\{\cos kx, \sin kx, \frac{1}{2}\}$ $k = 1, 2, 3, \dots$ is a complete orthogonal system in $L^2(-\pi;\pi)$. And, of course, also $\{e^{ikx}\}_{k\in\mathbb{Z}}$.

Parseval $\{e^{ikx}\}$ $\|f\|_2^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{+\infty} |c_k|^2 \underbrace{\|\phi_k\|_2^2}_{2\pi \ \forall k}$

i.e. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{n=-\infty}^{+\infty}|c_n|^2$

Parseval $\{\cos nx, \sin nx, \frac{1}{2}\}$

We proved that $\qquad b_0=0$

$$c_n = \frac{a_n - ib_n}{2} \quad \text{for } n \geq 0 \quad \left(c_0 = a_0/2\right)$$

$$c_n = \frac{a_n + ib_n}{2} \quad \text{for } n < 0$$

$$a_n = c_n + c_{-n} \quad \text{for } n \geq 0 \quad (a_0 = 2c_0)$$

$$b_n = i(c_n - c_{-n}) \quad \text{for } n > 0 \quad (b_0 = 0)$$

$$\sum_{n=-\infty}^{+\infty}|c_n|^2 = \sum_{n=1}^{\infty}|c_{-n}|^2 + |c_0|^2 + \sum_{n=1}^{\infty}|c_n|^2 = \text{we substitute the previous relationships} =$$

$$= \sum_{n=1}^{\infty}\frac{|a_n+ib_n|^2}{4} + \frac{|a_0|^2}{4} + \sum_{n=1}^{\infty}\frac{|a_n-ib_n|^2}{4} =$$

by def. of complex number modulus;
$z = x+iy$
$|z| = \sqrt{x^2+y^2}$

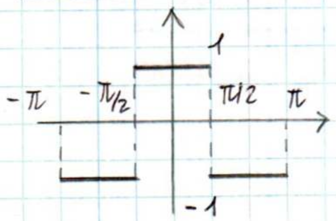
$$= \frac{|a_0|^2}{4} + \frac{1}{2}\sum\left(|a_n|^2+|b_n|^2\right), \text{ because } \frac{1}{4}\left[(a_n+ib_n)(\bar{a}_n - i\bar{b}_n) + (a_n-ib_n)(\bar{a}_n+i\bar{b}_n)\right] =$$

$$= \frac{1}{4}\left[|a_n|^2 - \cancel{i a_n \bar{b}_n} + \cancel{i b_n \bar{a}_n} + |b_n|^2 + |a_n|^2 + \cancel{i a_n \bar{b}_n} - \cancel{i b_n \bar{a}_n} + |b_n|^2\right] = \frac{1}{2}\left(|a_n|^2+|b_n|^2\right)$$

(Real) Parseval $\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right)}$

example

F.S. $f(x) = \frac{4}{\pi}\left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots\right)$



square wave

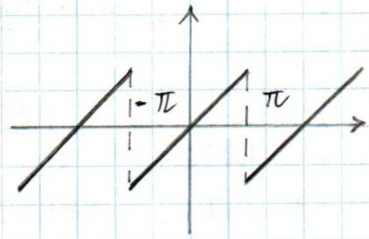
Parseval $\begin{pmatrix} b_n = 0 \\ a_0 = 0 \end{pmatrix}$ $\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx =$

$$= \underbrace{\frac{1}{\cancel{2}}\,\frac{\cancel{16}\,8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots\right)}_{\frac{1}{2}\sum_{1}^{\infty}|a_n|^2}$$

(where $a_n = \frac{(-1)^{n+1}}{\dots}\cdot\frac{4}{\pi}$

$$\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, dx}_{=1} \quad \Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots = \frac{\pi^2}{8}$$

Other ex. sawtooth



$$f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} 2^2 \frac{1}{n^2}$$

$$\frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{4}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)$$

$$\frac{1}{4\pi}\left(\frac{\pi^3-(-\pi)^3}{3}\right) = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$$

Theorem **Schwarz Inequality** (Cauchy Schwarz Buniakowskii)

$$\boxed{|(f,g)| \leqslant \|f\|_2 \, \|g\|_2}$$ (scalar product)

i.e. $$\left|\int_a^b f(x)\overline{g(x)}\,dx\right| \leqslant \left(\int_a^b |f(x)|^2 dx\right)^{1/2}\left(\int_a^b |g(x)|^2 dx\right)^{1/2}$$

N.B. this inequality is "obvious" if we take for granted the euclidean (∞ dimensional) nature of scalar products and L^2 norms, otherwise not so obvious...

Remark The Schwarz Inequality is also true if our scalar product contains a weight function $w(x) > 0$

i.e. $$(f,g) \overset{\text{def.}}{=} \int_a^b f(x)\overline{g(x)}\,\underbrace{w(x)}_{\text{in particular } w(x)=1}\,dx$$

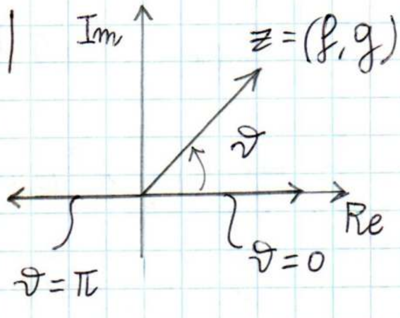
$$\left|\int_a^b f(x)\,\overline{g(x)}\,W(x)\,dx\right| \leq \left(\int_a^b |f(x)|^2 W(x)\,dx\right)^{1/2} \left(\int_a^b |g(x)|^2 W(x)\,dx\right)^{1/2}$$

Proof: set $\vartheta = \mathrm{Arg}(f,g)$

[N.B. if (f,g) is real $\quad \vartheta = 0$ if $(f,g) > 0$, $\quad \vartheta = \pi$ if $(f,g) < 0$]

Then $\underbrace{e^{-i\vartheta}}_{\text{produces a rotation } \vartheta}(f,g) = (e^{-i\vartheta} f, g) \equiv |(f,g)|$

coincides with the abs. val. in this case $\vartheta = 0; \pi$



$z_1 = \rho_1 e^{i\vartheta_1}$

$z_2 = \rho_2 e^{i\vartheta_2}$

$z_1 z_2 = \rho_1 \rho_2 e^{i(\vartheta_1 + \vartheta_2)}$

DE MOIVRE FORMULA

Now, $e^{-i\vartheta}$ is a complex number of argument $-\vartheta$ and $|e^{-i\vartheta}| = 1$

So the effect of multiplying $z \in \mathbb{C}$ by $e^{-i\vartheta}$ is to rotate the vector z (in the complex plane) of an angle $-\vartheta$

If $\mathrm{Arg}\, z = \vartheta$, then $e^{-i\vartheta} z = |z|$

Also, in this case, $(\vartheta = 0; \pi)$

$$(f e^{-i\vartheta}, g) = \mathrm{Re}\,(f e^{-i\vartheta}, g) = |(f,g)|$$

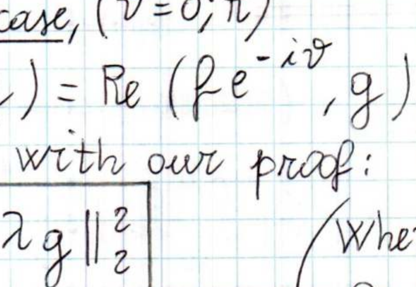
Now we go on with our proof:

$$\boxed{0 \leq \| f e^{-i\vartheta} + \lambda g \|_2^2} \qquad \begin{pmatrix} \text{Where } \lambda \in \mathbb{R} \\ \vartheta = \mathrm{Arg}(f,g) \end{pmatrix}$$

$= (f e^{-i\vartheta} + \lambda g, f e^{-i\vartheta} + \lambda g) =$ distributive property $=$

$= (f e^{-i\vartheta}, f e^{-i\vartheta}) + \lambda (f e^{-i\vartheta}, g) + \lambda (g, f e^{-i\vartheta}) + \lambda^2 (g,g) =$

$= \| f e^{-i\vartheta} \|_2^2 + \lambda 2\,\mathrm{Re}\,(f e^{-i\vartheta}, g) + \lambda^2 \| g \|_2^2 =$



$$= \|f\|_2^2 + 2\lambda |(f,g)| + \lambda^2 \|g\|_2^2$$

$0 \leq$ polynomial of degree 2 in the variable λ

(N.B. coeff. of λ^2 is $\|g\|_2^2 > 0$) $\Rightarrow \Delta \leq 0$

$$\frac{\Delta}{4} = \frac{b^2 - 4ac}{4} = \left(\frac{b}{2}\right)^2 - ac$$

$$\Rightarrow |(f,g)|^2 - \|f\|_2^2 \|g\|_2^2 \leq 0$$

$$|(f,g)|^2 \leq \|f\|_2^2 \|g\|_2^2$$

$$|(f,g)| \leq \|f\|_2 \|g\|_2 \qquad \text{(Q E D)}$$

$|(f,g)| \leq \|f\|_2 \|g\|_2$ True also on $L^2(\mathbb{R})$ or $L^2([0,+\infty))$, so this inequality is relevant also for the Fourier Transform and Laplace Transform.

<u>Lemma</u> 2.4.2

(p.36)

$$\boxed{(f,g) = \int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx = \sum_{\kappa \in I} c_\kappa \overline{d}_\kappa \|\phi_\kappa\|_2^2}$$

where c_κ and d_κ are the Fourier coeff. of f and g with respect to the orthogonal system $\{\phi_\kappa(x)\}$

$$f \sim \sum_{\kappa \in I} c_\kappa \phi_\kappa(x) \quad \text{with } c_\kappa = \frac{1}{\|\phi_\kappa\|_2^2}(f,\phi_\kappa)$$

and $g \sim \sum_{\kappa \in I} d_\kappa \phi_\kappa(x)$

$d_\kappa = \ldots$

This formula (Parseval-Plancherel) in the special case $f=g$ gives $\int_a^b |f(x)|^2\,dx = \sum_{\varkappa\in I} |c_\varkappa|^2 \|\phi_\varkappa\|_2^2$

<u>Proof</u> $|(f-S_\varkappa f, g)| \leqslant \|f-S_\varkappa f\|_2 \|g\|_2 \to 0$ as $\varkappa\to\infty$

we start writing this SCHWARZ INEQUALITY (whose general form we have proven before)

$$S_n f = \sum_{\varkappa=-n}^{n} c_\varkappa \phi_\varkappa(x)$$

(because $\|g\|_2 > 0$ and $\lim_{\varkappa\to\infty} \|f-S_\varkappa f\|_2 = 0$)

$$(f,g) = (f-S_\varkappa f + S_\varkappa f, g) = \underbrace{(f-S_\varkappa f, g)}_{\emptyset \text{ as } \varkappa\to\infty} + (S_\varkappa f, g)$$

therefore

$$(f,g) = \lim_{\varkappa\to\infty} (S_\varkappa f, g) = \lim_{n\to\infty} \left(\sum_{\varkappa=-n}^{n} c_\varkappa \phi_\varkappa(x), g(x)\right) =$$

$$= \lim_{n\to\infty} \sum_{\varkappa=-n}^{n} c_\varkappa \underbrace{(\phi_\varkappa, g)}_{\overline{(g,\phi_\varkappa)}} = \lim_{n\to\infty} \sum_{\varkappa=-n}^{n} c_\varkappa \underbrace{\bar{d}_\varkappa \|\phi_\varkappa\|_2^2}_{\text{FROM THE DEFINITION OF THE GENERALIZED COEFF'S}} =$$

$$= \sum_{\varkappa=-\infty}^{+\infty} c_\varkappa \bar{d}_\varkappa \|\phi_\varkappa\|_2^2$$

(QED)

<u>Th.</u> (No proof, but easy consequence of what we have done today) if $f\in L^1(a,b)$ and its F.S. is $f \sim \sum_{\varkappa\in I} c_\varkappa \phi_\varkappa(x)$ suppose $[\alpha,\beta]\subset[a,b]$

then

$$\boxed{\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta \sum_{\varkappa\in I} c_\varkappa \phi_\varkappa\,dx = \sum_{\varkappa\in I} c_\varkappa \int_\alpha^\beta \phi_\varkappa(x)\,dx}$$

{ if $[\alpha,\beta]\subset[a,b]$ <u>finite</u> <u>intervals</u> and $f\in L^1([a,b])\supset L^2([a,b])$ then it's ok to exchange $\int_\alpha^\beta \dots$ with F. Series }

<u>Remark</u> In applications to ODE and PDE we would also like to use $\boxed{[f(x)]' = \left[\sum_{\varkappa\in I} c_\varkappa \phi_\varkappa(x)\right]' = \sum_{\varkappa\in I} c_\varkappa \phi'_\varkappa(x)}$

In general this is more delicate than exchanging F.S. and integrals ... <u>But</u> it's OK if $c_k \to 0$ "quickly enough" (equivalent to $f(x)$ being "smooth enough").

Some families of orthogonal systems $\{\phi_k(x)\}_{k \in I}$ which <u>are</u> <u>not</u> the classical $\{e^{ikx}\}_{k \in \mathbb{Z}}$ or the equivalent $\{\cos kx, \sin kx, \frac{1}{2}\}_{k=1,2,3,\dots}$

Orthogonal Polynomials (pg. 45-59)

<u>Idea</u>: on a finite interval $[a;b]$ the power functions $1, x, x^2, x^3, \dots$ are <u>linearly</u> <u>independent</u>, i.e.,

$$\sum_{k=0}^{n} a_k x^k = 0 \text{ for all } x \in [a;b]$$

$$\iff a_k = 0 \text{ for } k = 0, 1, 2, \dots, n$$

This fact is somehow connected to the fact that it is possible to approximate large spaces of functions using polynomials. For example if $f \in L^p([a;b])$ $\forall \varepsilon > 0$ $\exists n \equiv n_\varepsilon$ and a polynomial $p(x)$ of degree n such that

$$\| f(x) - p(x) \|_{L^p([a;b])} < \varepsilon \quad \left\{ \begin{array}{l} \underline{\text{N.B.}} \text{ the } L^\infty \text{ case is} \\ \text{UNIFORM approximation} \end{array} \right\}$$

Given a scalar product on $L^2([a;b], w)$

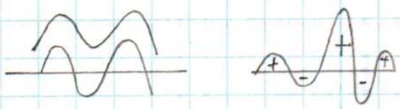
$$(f, g) \stackrel{\text{def}}{=} \int_a^b f(x) \overline{g(x)} \, w(x) \, dx$$

usually is not true that the power functions are orthogonal with respect to each other.

<u>BUT</u> it is possible to find a sequence of polynomials

$p_0(x), p_1(x), p_2(x), p_3(x), \dots,$ where each $p_k(x)$

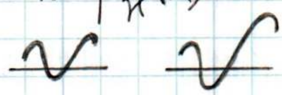
is a polynomial of (exact) degree n (the coeff. of x^n in $p_n(x)$ is $\neq 0$) such that $(p_n(x), p_j(x)) = \int_a^b p_n(x)\, \overline{p_j(x)}\, w(x)\, dx =$



$$= \begin{cases} 0 & \text{if } n \neq j \\ \alpha_n > 0 & \text{if } n = j \end{cases}$$

Furthermore if we fix $[a;b]$ and a weight $w(x) > 0$ for $x \in [a;b]$ (possibly $w(x) = 1$), the sequence $p_0(x), p_1(x), p_2(x), \dots$ is essentially UNIQUE.

"essentially" in the sense that each $p_n(x)$ can be dilated changing it into $\beta_n\, p_n(x)$



If we normalize, e.g. choosing $\|p_n\|_2 = 1 \;\; \forall n = 0, 1, 2, \dots$

then the sequence is unique. (other normalizations are possible, e.g., $p_n(0) = 1$)

Problem Given $[a;b]$ and $w(x)$, how do we compute $p_0(x), p_1(x), p_2(x) \dots$ orthogonal polynomials with respect to $(f, g) = \int_a^b f(x)\, \overline{g(x)}\, w(x)\, dx$?

Several answers:

1^{st}: "Bulldozer" answer: Gram-Schmidt Orthogonalization

Advantage: it "always" works (for any given $[a;b]$ $w(x)$) starting from the non-orthogonal but independent system $1, x, x^2, x^3, \dots$

Disadvantage: boring, inefficient, slow (even on computers)

2^{nd} possibility: It can be proven that all orthogonal polynomials satisfy a 3 terms recursive formula:

$$p_n \overset{*}{=} A_n\, p_{n-1}(x) + B_n\, p_{n-2}(x)$$

<u>Recipe</u>: compute $p_0(x)$ and $p_1(x)$. From p_0 and $p_1(x)$ compute $p_2(x)$ using $*$

From p_1 and p_2 compute p_3 etc....

<u>3rd method</u>: <u>Direct Formulae</u> given n produce p_n

<u>4th method</u>: <u>Semidirect Formulae</u> <u>(Rodriguez)</u>

$$p_n(x) = \frac{d^n}{dx^n}(\text{something})$$

<u>Other methods</u>...

Depending on $[a;b]$ and $w(x)$ the "smart" method could change.

In this course we'll see only a small sample of these methods (mostly following the book)

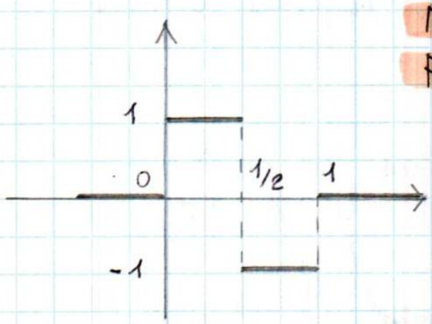
<u>Wavelets</u> (in particular <u>Haar</u> Basis) [ALSO P.62]

~1901

<u>Theorem</u> (<u>Haar</u>)

$$\psi(x) = \begin{cases} 1 & \text{if } x \in (0, \frac{1}{2}) \\ -1 & \text{if } x \in (\frac{1}{2}, 1) \\ 0 & \text{elsewhere} \end{cases}$$

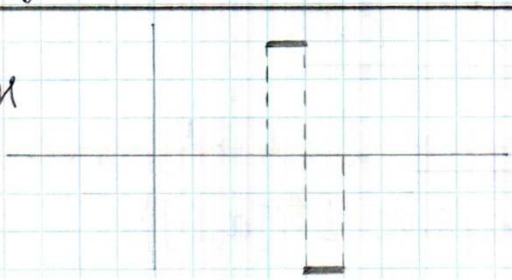
MOTHER FUNCTION



(ψ is "one tooth" of a square wave)

<u>Def.</u> $\boxed{\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k) \text{ for } j \in \mathbb{Z} \quad k \in \mathbb{Z}}$

ψ_{jk}



$\psi_{jk}(x)$ is a <u>complete</u> <u>orthonormal</u> <u>system</u> in $L^2(\mathbb{R})$

in other words any $f \in L^2(\mathbb{R})$ can be written

$$f(x) = \sum_{j,k} c_{jk} \psi_{jk}(x), \text{ where the "Haar coefficients"}$$

(Generalized F.S.)

$$c_{jk} = (f, \psi_{jk}) =$$

$$= \int_{-\infty}^{+\infty} f(x) \overline{\psi_{jk}(x)}\, dx = \int f(x)\, \psi_{jk}(x)\, dx$$

(real function) — interval support of ψ_{jk}

(N.B. $\|\psi_{jk}\|_2^2 = 1$ because $\{\psi_{jk}\}$ is an orthonormal system)

Advantage of Haar system: very simple $\psi(x)$

Disadvantages: many coefficients are needed to get a good approximation of $f(x)$. Properties of $f(x)$ (like smoothness) are mapped into c_{jk} in a complicated way.

Modern approach: choose better $\psi(x)$ mother functions such that $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$ for $j, k \in \mathbb{Z}$ is a complete orthonormal system in $L^2(\mathbb{R})$.

Other big families of orthogonal systems are given by eigenfunctions of specific operators.

COMPARE PAGE 41R & 44

Def. If A is an $n \times n$ matrix it defines linear operator: $\mathbb{R}^n \to \mathbb{R}^n$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

α is an EIGENVALUE and $\vec{x}$ is an EIGENVECTOR of this operator if $\boxed{A\vec{x} = \alpha\vec{x}}$

Def. Given a linear operator $\mathcal{L}: A \to A$, where A is a space of functions (e.g. $L^p([a,b])$ ...)

Linear $\boxed{\mathcal{L}(\alpha f(x)+\beta g(x)) = \alpha \mathcal{L}(f(x)) + \beta \mathcal{L}(g(x))}$

(ex. of linear operators: derivative, n-th derivative, Fourier transform, Laplace Transform ...)

If $\boxed{\mathcal{L}(f(x)) = \alpha \cdot f(x)}$, then we say that α is an EIGENVALUE and f is an EIGENFUNCTION of $\mathcal{L}$.

ex. $f(x) = e^{\alpha x}$ $\quad \mathcal{L}$ = derivative

$$\mathcal{L}(e^{\alpha x}) = \alpha e^{\alpha x} \qquad \text{eigenvalue} = \alpha$$

$\mathcal{L}$ = second derivative $\quad f(x) = \cos x, \sin x$ eigenfunctions

$(\sin x)'' = -\sin x \qquad \mathcal{L}(\sin x) = (-1)\sin x$

eigenvalue $= -1$

$\mathcal{L}(\cos x) = -\cos x \qquad$ eigenvalue $= -1$

N.B. $\mathcal{L}(3\cos x) = 3(-\cos x) = (-1)(3\cos x)$

same eigenvalue

ex. $\mathcal{L}$ = Fourier Transform

$$\mathcal{L}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}\, e^{-t^2/4a}$$ (Gaussian function)

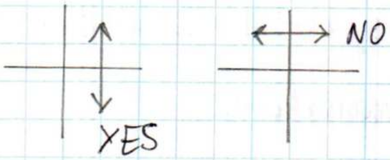
$a > 0 \qquad$ let's choose $a = \frac{1}{2}$

$$\mathcal{L}(e^{-\frac{1}{2}x^2}) = \sqrt{2\pi}\, e^{-\frac{1}{2}t^2}$$

so $f(x) = e^{-x^2/2}$ eigenfunction $\quad$ eigenvalue $= \sqrt{2\pi}$

Rem.1: we have to choose $a = \frac{1}{2}$ to have $f(x)$ eigenfunction of the F.T.; the corresponding eigenvalue is $\sqrt{2\pi}$ BUT $\Rightarrow$

$\mathcal{F}\left(c\, e^{-\frac{1}{2}x^2}\right) = c\sqrt{2\pi}\, e^{-\frac{1}{2}\xi^2}$

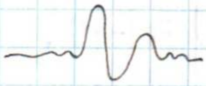


Rem. 2: (more details later) it is possible to find polynomials

P.43R $h_0(x), h_1(x), h_2(x), \dots$ $h_n(x)$ of degree n such that

$$F.T.\left(h_n(x)\, e^{-x^2/2}\right) = a_n \boxed{\underbrace{h_n}_{\text{HERMITE POLYNOMIALS}}(x)\, e^{-x^2/2}}$$ HERMITE FUNCTIONS

It is possible to prove also that these eigenfunctions of the F.T. $h_n(x)\, e^{-x^2/2}$ are a complete orthogonal system in $L^2(\mathbb{R})$.



LESS. 8 27/11/2008

Orthogonal polynomials

$p_n(x) = c_n x^n + \dots + c_1 x + c_0$ (linear combination of powers; degree $= n =$ highest power)

Given $[a; b]$ (also $[0; +\infty)$, also $(-\infty; +\infty)$) and a weight (i.e. a function $w(x) \geq 0$ for $x \in [a; b]$) we have a scalar product $(f, g) = \int_a^b f(x)\overline{g(x)}\, w(x)\, dx$; we also have an "essentially" unique sequence of polynomials of degree $n = 0, 1, 2, 3, \dots$ such that it is an orthogonal system, i.e. $(p_n, p_m) = \int_a^b p_n(x)\overline{p_m(x)}\, w(x)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \|p_n\|_2^2 & \text{if } n = m \end{cases}$

Depending on the weight choice we get different solutions.

---

case n. 1 **Legendre Polynomials**: $[a; b] = [-1; 1]$ $w(x) = 1$ $\forall x \in [-1$

our scalar product is $(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}\, dx$. They are:

$p_0(x) = 1$; $p_1(x) = x$; $p_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$; $p_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$;

$p_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

(are chosen in such a way that they make oscillation in $[-1; 1]$)

Let's check orthogonality in one case $(p_2, p_3) = \int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)dx$

$$= \int_{-1}^{1} \frac{15}{4}x^5 \underbrace{- \frac{9}{4}x^3 - \frac{5}{4}x^3}_{-\frac{14}{4}x^3} + \frac{3}{4}x\, dx = \left[\frac{\cancel{15}^{5}}{4}\frac{x^6}{\cancel{6}_2} - \frac{\cancel{14}^{7}}{4}\frac{x^4}{\cancel{4}_2} + \frac{3}{4}\frac{x^2}{2}\right]_{-1}^{1} =$$

$$= \frac{5}{8} - \frac{7}{8} + \frac{3}{8} - (\text{same things}) = 0$$

$(p_2, p_2) = \int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right)^2 dx > 0$

These are polynomials, but, in the interval they act, they are imitating sine & cosine (oscillation, cancellation effect...)
See figure 3.1, page 47 of the book.
To obtain O. P. we could apply the <u>Gram-Schmidt</u> orthogonalization to the sequence of powers $1, x, x^2, x^3, \ldots$
(N.B. these polynomials are linearly independent but they are not orthogonal w.r.t. $(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}\,dx$)

There are more efficient methods to obtain the same sequence of polynomials.
It can be proven that $\boxed{p_n(x) \overset{(*)}{=} \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n}$ (<u>Rodriguez</u>)

(hidden recursion...)
↳ because you can't compute directly the n-th derivative.

$p_0(x) = 1$ From the formula (*) (Rodriguez):

$$p_1(x) = \frac{1}{2^1 \cdot 1!} \frac{d}{dx} (x^2-1) = \frac{1}{2} 2x = x \quad ✓$$

$$p_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2} (x^2-1)^2 = \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) = \frac{1}{8} (4x^3 - 4x)' =$$

$$= \frac{1}{8} (12x^2 - 4) = \frac{3}{2} x^2 - \frac{1}{2} \quad ✓$$

<u>Theorem</u> (no proof in this course)

$$(☆) \quad \boxed{n\, p_n(x) - x(2n-1)\, p_{n-1}(x) + (n-1)\, p_{n-2}(x) = 0} \quad \text{(3-terms recursion)}$$

Special case of Theorem *: Almost all systems of O. P. satisfy a 3 terms recursion (o|||o)

From (☆) if we know the first two cases $p_0(x) = 1$, $p_1(x) = x$ then recursively we can compute $p_2(x)$, $p_3(x)$, ...

Legendre Polynomials

Rodriguez $\quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2-1) = x$$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2-1)^2 = \frac{1}{8} \frac{d}{dx} [2(x^2-1) \cdot 2x] =$$

$$= \frac{1}{8} [4x^3 - 4x]' = \frac{1}{8} [12x^2 - 4] = \frac{3}{2} x^2 - \frac{1}{2}$$

$$P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2-1)^3 = \frac{1}{8 \cdot 6} \frac{d^2}{dx^2} [3(x^2-1)^2 \cdot 2x] =$$

$$(x^4+1-2x^2)6x$$

$$(6x^5+6x-12x^3)$$

$$= \frac{1}{48} \frac{d}{dx} (30x^4 + 6 - 36x^2) =$$

$$= \frac{1}{48} (120x^3 - 72x) = \frac{5}{2} x^3 - \frac{3}{2} x$$

$$P_4(x) = \frac{1}{2^4 \cdot 4!} \frac{d^4}{dx^4} (x^2-1)^4 = \frac{1}{384} \frac{d^3}{dx^3} [8x (x^2-1)^3] =$$

$$= \frac{1}{384} \frac{d^2}{dx^2} [8(x^2-1)^3 + \underbrace{8x \cdot 3(x^2-1)^2 \cdot 2x}_{48x^2(x^2-1)^2}] =$$

$$= \frac{1}{384} \frac{d}{dx} [\underbrace{24(x^2-1)^2 \cdot 2x + 96x(x^2-1)^2}_{144x(x^2-1)^2} + \underbrace{48x^2\, 2(x^2-1) \cdot 2x}_{192x^3(x^2-1)}] =$$

$$= \frac{1}{384} \cdot [144(x^2-1)^2 + 144x \cdot 2(x^2-1)2x + 576x^2(x^2-1) +$$

$$+ 192x^3 \cdot 2x] =$$

$$= \frac{1}{384} [\underbrace{144x^4} + 144 - 288x^2 \underbrace{+576x^4} - 576x^2 \underbrace{+576x^4} - 576x^2 + \underbrace{384x^4}]$$

$$= \frac{1}{384} [1680x^4 - 1440x^2 + 144] = \frac{35}{8} x^4 - \frac{15}{4} x^2 + 3/8$$

3 term recursion

$$n\, p_n(x) - x(2n-1)\,p_{n-1}(x) + (n-1)\,p_{n-2}(x) = 0$$

$$p_0(x) = 1 \qquad p_1(x) = x$$

$$2\, p_2 - x(2\cdot 2-1)\, x + (2-1)\, 1 = 0$$

$$2 p_2 = 3x^2 - 1 \qquad p_2 = \frac{3x^2}{2} - \frac{1}{2}$$

$$3 p_3 - x(6-1)\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + (3-1)\, x = 0$$

$$3 p_3 = \frac{15}{2}x^3 - \frac{5}{2}x - 2x = \frac{15}{2}x^3 - \frac{9}{2}x$$

$$p_3 = \frac{5}{2}x^3 - \frac{3}{2}x$$

$$4 p_4 - x(8-1)\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) + 3\left(\frac{3x^2}{2} - \frac{1}{2}\right) = 0$$

$$4 p_4 - \frac{35}{2}x^4 + \frac{21}{2}x^2 + \frac{9}{2}x^2 - \frac{3}{2} = 0$$

$$4 p_4 = \frac{35}{2}x^4 - 15x^2 + \frac{3}{2}$$

$$p_4 = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$

HIGHLY MORE EFFECTIVE PROCEDURE

Exercise: compute the first 5 or 6 (or 7) Legendre Polynomials both with Rodriguez Formula and the (✱) recursion comparing efficiency.

A little detour about recursive method: for example, Fibonacci Numbers $c_0, c_1, c_2, c_3, \dots$

$$c_0 = c_1 = 1 \quad \text{and} \quad c_n = c_{n-1} + c_{n-2}$$

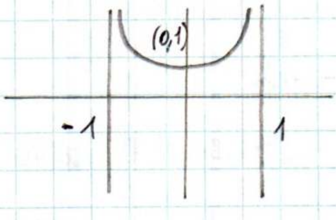
$$c_2 = 1+1 = 2 \quad c_3 = 2+1 = 3 \quad c_4 = \dots$$

exercise: either on your own (better) or cheating (Google) find a direct closed formula for the n-th Fibonacci Number c_n (hint: it is the linear combination of two exponentials one increasing to $\to\infty$ and the other one decreasing to 0) (hint: Golden Ratio)

Sometimes (but not always) it is possible, like in the case of Fibonacci, to obtain a "closed formula" from the recursive definition.

Case 2: Chebycheff Polynomials $T_n(x)$ orthogonal system of polynomials in $(-1;1)$ with the weight

$$W(x) = \frac{1}{\sqrt{1-x^2}}$$



$$(f,g) = \int_{-1}^{1} f(x)\, \overline{g(x)}\, \frac{1}{\sqrt{1-x^2}}\, dx \quad (✱✱)$$

$T_0(x) = 1 \qquad T_1(x) = x \qquad T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x \qquad T_4(x) = 8x^4 - 8x^2 + 1 \qquad T_5(x) = \dots$

First method to compute these polynomials is Gram-Schmidt apply to $1, x, x^2, x^3, \dots$ w.r.t. (✱✱)

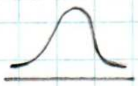
2nd method $\boxed{T_{n+1}(x) - 2x\,T_n(x) + T_{n-1}(x) = 0}$ (try it)

$$T_2(x) - 2x\,T_1(x) + T_0(x) = 0$$

Theorem "closed formula" for Chebycheff Polynomials

$$T_n(x) = \cos(n \arccos x) = \ldots =$$
$$= x^n - \binom{n}{2} x^{n-2}(1-x^2) + \binom{n}{4} x^{n-4}(1-x^2)^2 - \ldots$$

Case 3: Hermite Polynomials (and Hermite Functions)

Consider the interval $(-\infty; +\infty) \equiv \mathbb{R}$ $\quad w(x) = e^{-x^2}$ 

$$(f,g) = \int_{-\infty}^{+\infty} f(x)\,\overline{g(x)}\, e^{-x^2} dx$$

We get Hermite Polynomials $H_n(x)$

$\boxed{H_{n+1}(x) - 2x\,H_n(x) + 2n\,H_{n-1}(x) = 0}$ $\quad H_0(x) = 1 \quad H_1(x) = 2x$

Rodriguez: $\boxed{H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n} e^{-x^2}}$

e.g. $H_1(x) = (-1)^1 e^{x^2} \dfrac{d}{dx}(e^{-x^2}) = -e^{x^2}(-2x)e^{-x^2} = 2x$

$$H_2(x) = (-1)^2 e^{x^2} \frac{d}{dx}(-2x e^{-x^2}) = e^{x^2}(-2e^{-x^2} - 2x(-2x)e^{-x^2})$$
$$= -2 + 4x^2$$

Important Remark Once we have generated the sequence $H_0(x), H_1(x), H_2(x), \ldots$ of Hermite Polynomials we know that

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2} dx = \begin{cases} 0 & n \neq m \\ \|H_n\|_2^2 > 0 & \text{if } n = m \end{cases}$$

We can also consider the $\boxed{h_n(x) = H_n(x)\, e^{-x^2/2}}$ called Hermite Functions (N.B. these functions are not polynomials, but they are polynomials times the square root of a weight)

$$h_0(x) = e^{-x^2/2}\ ;\ h_1(x) = 2x\,e^{-x^2/2}\ ;\ h_2(x) = (4x^2 - 2)\,e^{-x^2/2}$$

...