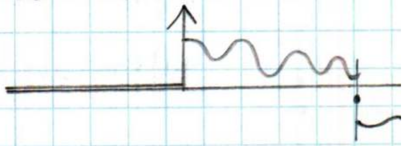
Theorem (No proof) The functions $h_n(x) = H_n(x)\, e^{-x^2/2}$ for $n = 0, 1, 2, \dots$ are <u>all</u> the eigenfunctions of the Fourier Transform. = complete orthogonal system (Diagonalization of F.T. in this ∞-dim. space).

LAPLACE TRANSFORM (Chapter V)

It is a sort of Fourier Transform on a half-line $[0; +\infty)$.

Def. We say that $\boxed{f(t) \in E}$ if $f(t) = 0$ for $t < 0$ and $\exists\, \alpha \in \mathbb{R}$ such that



(1) $f(t)\, e^{-\alpha t}$ in $L^1([0; +\infty))$; piecewise

(2) $C^1(\mathbb{R})$; (3) $f(x) = \dfrac{f(x^+) + f(x^-)}{2}$ for each real number x.

Def.
$$\boxed{\begin{aligned} \mathcal{L}[f](s) &= \int_0^{+\infty} f(t)\, e^{-st}\, dt \\ (s \in \mathbb{C}) \quad &= \int_0^{+\infty} f(t)\, e^{-\sigma t} (\cos \omega t - i \sin \omega t)\, dt \end{aligned}}$$

Tradition $\boxed{s = \sigma + i\omega} \in \mathbb{C}$ $\quad (\sigma = \mathrm{Re}(s)\,;\ \omega = \mathrm{Im}(s))$

ex. $f(t) = e^{at}$ where $a = b + ic \in \mathbb{C}$

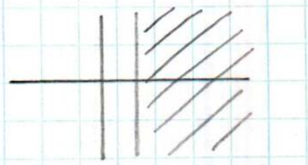
$\boxed{\mathcal{L}(f) = F(s)}$ (Notation)

The original function has real variable, the transform a complex on

$$\mathcal{L}(f) = F(s) = \int_0^{\infty} e^{at} e^{-st}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt$$

It's easy to see that this integral converges $\iff \mathrm{Re}(s) > \mathrm{Re}$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_{t=0}^{t=+\infty} = \frac{1}{s-a}$$



Remarks The original function $f(t)=e^{at}$ is $f:\mathbb{R}\to\mathbb{C}$
(in particular is a function of a real variable t)
(also $f(t)$ itself could be real).

The transform function $F(s)$ is a $F:\mathbb{C}\to\mathbb{C}$
This function is also holomorphic except for a finite number of singularities.

In the case of this example, the only singularity is at $s=a$ (simple pole, where DEN $=0$)

Holomorphic: locally can be written with a complex power series (Taylor Series) with radius $R>0$;
equivalentely, the complex derivative exists finite

$$\lim_{h\to 0}\frac{F(s+h)-F(s)}{h}=F'(s)$$

Analytic continuation: the integral that defines the holomorphic function $F(s)$ is well defined only in a half-plane $(Re(s)>Re(a))$ BUT the function $F(s)$ itself can be (easily or not, depending on cases) analytically continued to all $\mathbb{C}\setminus\{\text{finite set of singularities}\}$
This pattern, that we have already seen in the simple case $f(t)=e^{at}$, actually holds in general if $f(t)\in E$.

Laplace Transform

Linearity $\mathcal{L}(\alpha f(t)+\beta g(t))=\alpha\mathcal{L}(f)_{(s)}+\beta\mathcal{L}(g)_{(s)}=$

$$=\alpha F(s)+\beta G(s)$$

ex.1) using linearity compute $\mathcal{L}(\sin at)$ and exponentials

N.B. When we say $\mathcal{L}(\sin at)$ what we mean is $\mathcal{L}(f(t))$ where $f(t) = \begin{cases} 0 & \text{if } t<0 \\ \sin at & \text{if } t \geq 0 \end{cases}$

$$\sin at \overset{\text{Euler's}}{=\!=} \frac{e^{iat} - e^{-iat}}{2i}$$

We use the previous: $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

$$\mathcal{L}(\sin at) = \frac{1}{2i}\mathcal{L}(e^{iat}) - \frac{1}{2i}\mathcal{L}(e^{-iat})$$

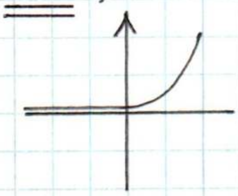
$$= \frac{1}{2i}\left\{\frac{1}{s-ai} - \frac{1}{s+ai}\right\} = \frac{1}{2i}\,\frac{\cancel{s}+ia-(\cancel{s}-ia)}{(s^2-(ia)^2)} = \frac{2ia}{2i(s^2+a^2)} =$$

$$= \frac{a}{s^2+a^2}$$

ex.2) $$\mathcal{L}(\cos at) \overset{\text{Euler's}}{=\!=} \mathcal{L}\left(\frac{e^{iat}+e^{-iat}}{2}\right) = \frac{1}{2}\left\{\frac{1}{s-ia} + \frac{1}{s+ia}\right\} =$$

$$= \frac{1}{2}\,\frac{s+\cancel{ia}+(s-\cancel{ia})}{(s^2+a^2)} = \frac{s}{s^2+a^2}$$

ex. 3)



$$\mathcal{L}(t^2) = \int_0^{+\infty} t^2 e^{-st}\,dt \overset{\text{b.p.}}{=\!=}$$

$$f(t) = \begin{cases} 0 & \text{for } t<0 \\ t^2 & \text{for } t \geq 0 \end{cases}$$

$$= \left[\frac{t^2 e^{-st}}{-s}\right]_0^{+\infty} + \int_0^{+\infty} \frac{e^{-st}}{+s}\,2t\,dt =$$

$$= 0 + \frac{2}{s}\int_0^{+\infty} t\,\underbrace{e^{-st}\,dt}_{d\left(\frac{e^{-st}}{-s}\right)} = +\frac{2}{s}\Bigg\{\underbrace{\left[t\,\frac{e^{-st}}{-s}\right]_0^{+\infty}}_{0} +$$

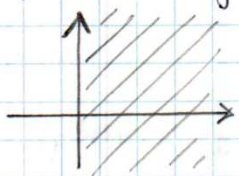
$$- \int_0^{+\infty} \frac{e^{-st}}{-s}\,dt\Bigg\} = +\frac{2}{s^2}\int_0^{+\infty} e^{-st}\,dt =$$

$$= +\frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^{+\infty} = \frac{2}{s^3}$$

More generally, using n integrations by parts, we get $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$
Put this in a table and then by linearity is easy to do the $\mathcal{L}$(polynomials). Actually using Γ function we can also compute $\mathcal{L}(t^{\alpha})$, $\alpha \in \mathbb{R}$

Def. $\boxed{\Gamma(\alpha+1) = \int_0^{+\infty} t^{\alpha} e^{-t} dt}$ Euler's Gamma Function

Theorem $\Gamma(n+1) = n!$ if $n = 0, 1, 2, \ldots$ But $\Gamma(x)$ is actually well defined for $x \in \mathbb{R}$; even more it is an holomorphic function of $z \in \mathbb{C}$, $\Gamma(z)$. There are singularities, but they are in the negative half-plane $Re(z) < 0$



Punchline (clou)

$$\boxed{\mathcal{L}(t^{\alpha}) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}}$$

ex. $\mathcal{L}(\sqrt{t}) = \mathcal{L}(t^{1/2}) = \dfrac{\Gamma(3/2)}{s^{3/2}} = \dfrac{\alpha}{\sqrt{s^3}}$

Where $\alpha = \int_0^{+\infty} t^{1/2} e^{-t} dt$

PROPERTIES of L.T.

$$\boxed{\mathcal{L}[e^{at} f(t)](s) = F(s-a)}$$

$$\boxed{\mathcal{L}[f(\alpha t)](s) = \frac{1}{\alpha} F(s/\alpha)}$$

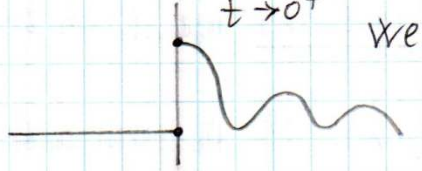
$$\boxed{\mathcal{L}(f(t-T)) = e^{-sT} F(s)}$$

$$\boxed{\mathcal{L}(t^n f(t))(s) = (-1)^n F^{(n)}(s)}$$

Derivative property (very important)

$\lim_{t\to 0^-} f(t)=0$ because $f(t)=0$ if $t<0$

$\lim_{t\to 0^+} f(t)$ can be $\alpha \neq 0$ We call $\alpha = f(0^+)$



$$(\star)\quad \boxed{\mathcal{L}[f'(t)] = sF(s) - f(0^+)}$$

Proof

$$\mathcal{L}(f'(t))_{(s)} = \int_0^{+\infty} f'(t)e^{-st}\,dt =$$

$$= \left[f(t)e^{-st}\right]_{t=0}^{t=+\infty} + s\int_0^{+\infty} f(t)e^{-st}\,dt = sF(s) - f(0^+)$$

$$(\star\star)\quad \boxed{\mathcal{L}(f^{(n)}(t))(s) = \underbrace{s^n F(s)}_{\substack{\text{MAIN TERM} \\ \text{(like F.T.)}}} - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - \dots - f^{(n-1)}(0^+)}$$

Remark Because of this derivative property, the Laplace Tranfo[rm] $\mathcal{L}$ is well-adapted to solving ODE linear order n with a Cauchy problem with initial conditions in 0.

$$(*)\quad a_n y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f(t) \quad t\in(a,b)\subset\mathbb{R}$$

Cauchy's n initial conditions $\begin{cases} \text{looking for } y(t) \text{ satisfying } (*) \text{ and} \\ y(0)=c_0 \quad y'(0)=c_1 \quad y''(0)=c_2 \dots y^{(n-1)}(0)=c_{n-1} \end{cases}$

Procedure: take Laplace Tranform of both sides of $(*)$,

get $Y(s)\cdot\underbrace{(\text{polynomial in } s)}_{P(s)} = F(s)$

$\Rightarrow Y(s) = \dfrac{F(s)}{P(s)}$ explicit formula for $\mathcal{L}(y(t))$, where we have already used the Cauchy n initial conditions.

Final Step $\mathcal{L}^{-1}\left(\dfrac{F(s)}{P(s)}\right) = y(t)$ unique solution

There is a general formula to compute $\mathcal{L}^{-1}$, but it involves line integrals in $\mathbb{C}$ (Key words: residues, contour integration, Cauchy Theorem):

Inversion Formula $\boxed{f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty} F(s) e^{st} ds}$

(N.B. in principle, you can invert any $F(s)$ holomorphic with this formula; we will not use it in this 5 cr. course)

If $F(s)$ is of a particular type (e.g. a rational function $\frac{p(s)}{q(s)}$ with p, q polynomials),

It is possible to compute $f(t) = \mathcal{L}^{-1}(F(s))$ using tables plus properties of $\mathcal{L}$.

We have defined on $\mathbb{R}$ $f * g_{(x)} = \int_{-\infty}^{+\infty} f(x-y) g(y) dy$

convolution of f and g.

In the F.T. theory, we have seen that $(f*g)^\wedge(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

In Laplace Transform theory we also have $\mathcal{L}[f*g]_{(s)} = F_{(s)} G_{(s)}$

but we need to make an observation, because $f_{(t)}$ and $g_{(t)}$ by assumption are $=0$ for $t<0$

$$\int_{-\infty}^{+\infty} f(x-y) \underbrace{g(y)}_{\text{this factor is } =0 \text{ when } y<0} dy = \int_{0}^{+\infty} \overbrace{f(x-y)}^{\text{now observe this other factor } =0 \text{ when } x-y<0 \text{ i.e. } x<y} g(y) dy =$$

$$= \int_0^{x} f(x-y) g(y) dy$$

Book, page 89 has a small table of Laplace Transform.

Google: table of Laplace Transforms

For future reference: there will be cases where we will need to compute $\mathcal{L}\left[\frac{p(s)}{q(s)}\right]$. It will be necessary to know the theory of partial fractions. There are different level of complexity depending on the denominator $q(s)$.

Solve $q(s) = a(s-s_1)(s-s_2)\cdots(s-s_n)$

4 levels:

(1) $q(s)=0$ has n solutions real and distinct:

$$\frac{p(s)}{q(s)} = \frac{A_1}{(s-s_1)} + \frac{A_2}{(s-s_2)} + \dots + \frac{A_n}{(s-s_n)}$$

(2) $q(s)=0$ has real sol's but there are multiplicities.

(3) $q(s)=0$ has simple pairs of conjugate complex sol's.
(remember that if $q(s)$ has real coeff's and complex roots, these roots appear in conjugate pairs)

(4) complex conjugate with multiplicities..

LESS. 9 11/12/2008

$$\boxed{\mathcal{L}\left(\frac{t^{n-1}e^{at}}{(n-1)!}\right) = \frac{1}{(s-a)^n}}$$

$a \in \mathbb{C}$ (complex constant)

becomes a complex translation

Proof we have proven already $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$$\mathcal{L}(t^{n-1}) = \frac{(n-1)!}{s^n} \qquad \mathcal{L}\left(\frac{1}{(n-1)!}t^{n-1}\right) = 1/s^n$$

by linearity

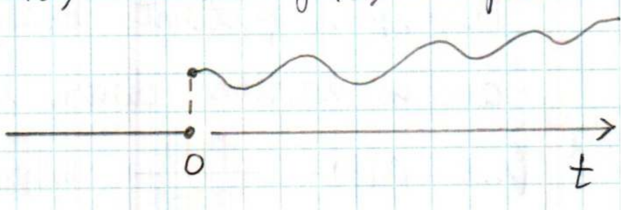
instead of s put $s-a$ and get the result

$\mathcal{L}(e^{at}f(t)) = F(s-a)$ (Q.E.D.)

translation property

Note the set E of Laplace-transformable functions satisfies:
$f(t): (-\infty;+\infty) \to \mathbb{C}$ (maybe $\mathbb{R}$) with $f(t)=0$ for $t<0$
$\mathbb{R}$

$\left\{\begin{array}{l} f(t) \text{ could be growing} \\ \text{BUT not too much} \end{array}\right\}$



Meaning of not growing too much :

$\exists\, \alpha \in \mathbb{R}$ such that $f(t)\, e^{-at} \in L^1(\mathbb{R})$ if $\mathrm{Re}(a) > \alpha$

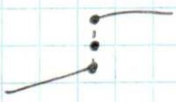
(for example $f(t) =$ polynomial are OK

$f(t) = e^{t^3}$ not OK)

$f(t)$ is <u>piecewise</u> $C^1(\mathbb{R})$; if there is a jump $f(x) = \dfrac{f(x^+) + f(x^-)}{2}$

{ i.e. there is only a finite number of discontinuities }

(the value you get is the midpoint of jump)



<u>Theorem</u> If both $f(t)$ and $p(t) = \int_0^t f(u)\, du$ are in E

then $\mathcal{L}[p(t)] = \frac{1}{s} F(s)$ (N.B. $p(0) = 0$)

$\mathcal{L}(f(t))_{(s)}$

<u>Proof</u>: $\mathcal{L}(p'(t)) = \underbrace{\mathcal{L}(f(t))}_{F(s)} = s\, P(s) - \cancel{p(0^+)}^{\,\emptyset}$ $\quad P(s) = \mathcal{L}(p(t))$

$F(s) = s\, P(s) \qquad \frac{1}{s} F(s) = P(s) \qquad$ (Q.E.D.)

<u>Theorem</u> (No proof) If $f(t)$ and $g(t) \in E$, then also $f * g_{(t)} \in E$

Also: $$f * g_{(t)} = \int_{-\infty}^{+\infty} f(y)\, g(t-y)\, dy =$$

$$= \int_0^t f(y)\, g(t-y)\, dy$$

<u>and</u> $\mathcal{L}(f * g) = \mathcal{L}(f)\, \mathcal{L}(g) = F(s)\, G(s)$

(convolutions appear frequently in resolution of D.E.'s)

Suppose $\underset{\mathcal{L}(g(t))}{\underset{|||}{G(s)}} = \sum_j \dfrac{A_j}{(s-a_j)^{n_j}}$ then $g(t) = \sum_j A_j \dfrac{t^{n_j - 1}\, e^{a_j t}}{(n_j - 1)!}$

(easy consequence of the property we've seen)

This is important because <u>proper rational functions</u> can be written down in this way (<u>partial fractions</u>).

(i.e. $g(t) = \dfrac{p(t)}{q(t)}$ where p, q polynomials and the degree of $p(t)$ is $<$ degree of $q(t)$)

As you know very well, any rational function is equal to a polynomial plus a proper rational function.

Example: $\frac{3t^3-t^2+1}{t^2-1}$ non-proper rational function

$$\begin{array}{r|l} 3t^3-t^2+1 & t^2-1 \\ \hline -3t^3 \quad +3t & 3t-1 \\ \hline -t^2+3t+1 & \\ +t^2 \quad -1 & \\ \hline 3t & \end{array}$$

REMAINDER

$$\frac{3t^3+1-t^2}{t^2-1} = \underbrace{\frac{3t}{t^2-1}}_{\text{proper rational function}} + 3t - 1$$

<u>Auxiliary theorem</u> (5.3.9) (DON'T USE WHEN YOU HAVE MULTIPLE ROOTS!)

If $G(s)$ is a <u>proper</u> (degree $P <$ degree Q) rational function $G(s) = \frac{P(s)}{Q(s)}$ (P, Q polynomials with <u>real coeff's</u>)

Call $g(t) = \mathcal{L}^{-1}(G(s))$; then (1) to a <u>simple</u> zero α of $Q(s)$ corresponds in $g(t)$ the (additive) term $\frac{P(\alpha)}{Q'(\alpha)} e^{\alpha t}$;

(2) to a <u>conjugate</u> <u>(simple)</u> <u>pair</u> $\alpha \pm i\omega$ of zeros of $Q(s)$ corresponds in $g(t)$ the term $e^{\alpha t}(A \cos \omega t - B \sin \omega t)$ where A and B are <u>real numbers</u> computable from

$$A + iB = 2\left[\frac{P(s)}{Q'(s)}\right]\Bigg|_{s = \alpha + i\omega}$$

(3) (almost obvious) If $Q(s) = Q_1(s)\, Q_2(s)$, with Q_1 and Q_2 polynomials, and if $Q_1(\alpha) = 0$, then

$$Q'(\alpha) = Q_1'(\alpha)\, Q_2(\alpha)$$

<u>Example</u>: $G(s) = \frac{s}{(s+3)(s^2+4s+5)}$ {would be more difficult if we had the 3rd degr. denominator not factorized

By the auxiliary th. to the zero $\alpha = -3$ of our denominator corresponds in $g(t) = \mathcal{L}^{-1}(G(s))$ the term $\frac{P(-3)}{Q'(-3)} e^{-3t} =$

$$= \frac{-3\,e^{-3t}}{[s^2+4s+5]\Big|_{s=-3}} = \frac{---}{9-12+5} = -\frac{3}{2}\,e^{-3t}$$

The polynomial s^2+4s+5 has zeros $-2 \pm \sqrt{2^2-5} = -2 \pm i$
(pair of conjugate non-real zeros) (case (2) of aux. th.)
then to them corresponds in $g(t)$ the term

$$e^{-2t}(A\cos t - B\sin t) \qquad \begin{pmatrix} \alpha = -2 \\ \omega = 1 \end{pmatrix}$$

Where $A + iB = 2\left[\frac{P(s)}{Q'(s)}\right]_{s=-2+i} = 2\left[\frac{s}{(s+3)(2s+4)}\right]_{s=-2+i} =$

$$= 2\left[\frac{-2+i}{(-2+i+3)(-4+2i+4)}\right] = 2\,\frac{(-2+i)(1-i)}{(1+i)\,2i\,(1-i)} = \frac{-1+3i}{2i}\,\frac{(-i)}{(-i)} =$$

$$= \frac{3+i}{2} = \frac{3}{2} + \frac{1}{2}i \Rightarrow \begin{matrix} A = 3/2 \\ B = 1/2 \end{matrix}$$

$$\Rightarrow g(t) = -\frac{3}{2}e^{-3t} + e^{-2t}\left(\frac{3}{2}\cos t - \frac{1}{2}\sin t\right)$$

this is the inverse Laplace transform computed
without the inversion formula (that involves
complex integrals ...).

Ex. 2)

$$\begin{cases} y'' - y = 2e^t \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$

ODE linear, 2^{nd} order, constant coeff's
not homogeneous
(N.B. the ODE by itself has ∞^2 sol.'s
depending on c_1 and c_2 arbitrary
constants BUT the 2 Cauchy
initial conditions make the sol. unique

WE WANT TO SOLVE
THIS PROBLEM USING
LAPLACE TRANSFORM

$$\mathcal{L}(y^{(n)}(t)) = s^n Y(s) - s^{n-1}y(0^+) - s^{n-2}y'(0^+) - s^{n-3}y''(0^+) - \dots$$

IN OUR CASE:

$$\underbrace{s^2 Y(s) - 2s - 1}_{} - \overbrace{Y(s)}^{\mathcal{L}(y(t))} = \left.\frac{2}{s-1}\right\}\mathcal{L}(2e^t)$$

N.B. this is a
1^{st} degree equation
in $Y(s)$

$$(z^2-1)\,Y(z) = \frac{2}{z-1} + 2z+1 = \frac{2(2z+1)(z-1)}{z-1}$$

$$Y(z) = \frac{2z^2-z+1}{(z+1)(z-1)^2} \underset{\substack{\text{partial}\\\text{fractions}}}{\overset{(*)}{=}} \frac{A}{z+1} + \overbrace{\frac{B}{z-1} + \frac{C}{(z-1)^2}}^{m=2}$$

proper rational function of z

In general, in the partial fractions expansions, to a real root α of multiplicities m correspond

$$\frac{B_1}{z-\alpha} + \frac{B_2}{(z-\alpha)^2} + \frac{B_3}{(z-\alpha)^3} + \dots + \frac{B_m}{(z-\alpha)^m}$$

The case of multiple roots is not included in the aux. th., but...

$(*)$ is an equality between rational functions, true $\forall z \in \mathbb{C} \setminus \{$roots of denom.$\}$. We can transform $(*)$ into $(**)$ an equality of polynomials:

$$2z^2-z+1 = A(z-1)^2 + B(z+1)(z-1) + C(z+1) \qquad (**)$$

One way to compute A, B, C would be to expand the 2nd degree polynomial on the r. h. s. of $(**)$ and equate coeff.'s; this algorithm is OK for degree 2 but it becomes worse and worse as the degree gets higher. That's why we'll see another algorithm.

$$(**) = A(z^2-2z+1) + B(z^2-1) + C(z+1) = (A+B)z^2 + (-2A+C)z + A-B+C$$

This gives a linear system:

$$\begin{cases} A+B=2 \\ -2A+C=-1 \\ A-B+C=1 \end{cases}$$

linear system of 3 eq's in 3 unknowns A, B, C

This method "sucks" if degrees are high (OK for 2×2 or 3×3 systems)

$$\begin{array}{r} 2A+2C=4 \\ -2A+C=-1 \\ \hline 3C=3 \end{array} \qquad -2A=-2,\; A=1 \qquad C=1$$

$$B = A+C-1 = 1+1-1 = 1$$

Other method:

Substitute in (**) the roots of the initial denominator:

$\underline{s=-1}$ $\quad 2(-1)^2-(-1)+1 = A(-2)^2 + \cancel{B\cdot 0} + \cancel{C\cdot 0}$

$$4 = 4A \Rightarrow A=1$$

$\underline{s=+1}$ $\quad 2-1+1 = C\,(1+1) \qquad 2 = 2C \Rightarrow C=1$

Because of the multiplicity $m=2$ we run out of values to use...

Use (***) which is the $\frac{d}{ds}$ of (**):

$$4s-1 = 2A(s-1)+2Bs+C = 2(s-1)+2Bs+1$$

$\underline{s=1}$ $\quad 4\cdot 1-1 = 2B+1 \qquad 4 = 2B+2 \qquad 2=2B$

$B=1$

To be complete, we should also give an algorithm to find the partial fraction constants when there are multiple $\alpha \pm i\beta$. WE SKIP IT.

We have $Y(s) = \frac{1}{s+1} + \frac{1}{s-1} + \frac{1}{(s-1)^2}$; now using tables

we get: $\quad y(t) = \mathcal{L}^{-1}(Y(s)) = \underbrace{\underbrace{e^{-t}+e^{t}}_{\text{hom.}} + \underbrace{te^{t}}_{\text{forcing term}}}_{\text{in the classical way}}$