Leysin Seminar

SHORT REMINDER of OUR ROADMAP

$(S) \begin{cases} \mathcal{L} u = f \text{ in } \Omega & (\text{e.g. } -u'' = f) \\ + \text{ B.C. on } \partial\Omega \end{cases}$ — $\mathcal{L}$: LINEAR DIFFERENTIAL OPERATOR — Strong or differential formulation of the problem

$\downarrow$

$(W) \begin{cases} ?\, u \in V \text{ s.t.} \\ a(u, v) = F(v), \quad \forall v \in V \end{cases}$ — PRATIK HA USATO (u', v') — Weak or integral formulation of the problem

$\downarrow$ choose a subspace $V_h \subset V$, $\dim V_h = N_h < +\infty$

$(G) \begin{cases} ?\, u_h \in V_h \text{ s.t.} \\ a(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h \end{cases}$ — Galerkin formulation

$\downarrow$ choose a basis for V_h $\rightsquigarrow$ NB: if the space is X_h^r (space of piecewise continuous polynomials of degree r) THEN we get the Galerkin - FEM.

(G-Alg) $\quad A\vec{u} = \vec{f}$

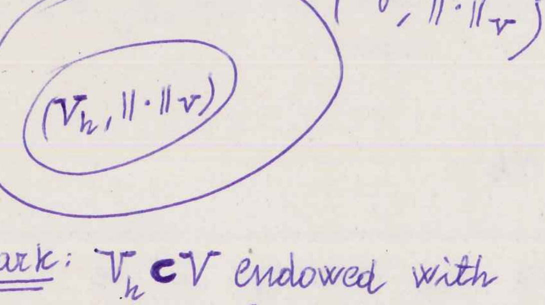
$(K\underline{u} = M\vec{u})$ $\quad A \in \mathbb{R}^{N_h \times N_h}$, stiffness matrix

$\boxed{1}$

(G) GALERKIN PROBLEM (go to a FINITE - dimensional space)

We choose a subspace $V_h \subset V$ s.t. $\dim V_h = N_h < +\infty$.

$(G) \begin{cases} ?\, u_h \in V_h \subset V \text{ s.t.} \\ a(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h \end{cases}$ $\qquad a: V \times V \to \mathbb{R}$, $\quad F: V \to \mathbb{R}$



Since V_h is a subspace of V, and V is a Hilbert space, V_h is ALSO a Hilbert space, THUS by means of Lax-Milgram Theorem we conclude that $\exists!$ solution $u_h \in V_h$ of (G) and we have the stability result: (: the sol. u_h is BOUNDED by the RHS)

Remark: $V_h \subset V$ endowed with the same norm (it is a subspace of V; IT INHERITS ALL THE FEATURES of V, BUT IT IS FINITE-DIMENSIONAL.)

$$\|u_h\|_V \leq \frac{1}{\alpha} \|F\|_{V'}$$

α: coercivity constant of $a(\cdot,\cdot)$

V': DUAL SPACE of V: space of linear & continuous functionals on V. $\|F\|_{V'} := \sup_{v \in V} \frac{|F(v)|}{\|v\|}$

$\Rightarrow \exists!\, \vec{u} \in \mathbb{R}^{N_h}: A\vec{u} = \vec{f}$ **because** $V_h \simeq \mathbb{R}^{N_h}$ {"EVERY FINITE DIMENSIONAL VECTOR SPACE IS ISOMORPHIC TO ITS SPACE of COORDINATES."

We have established EXISTENCE, UNIQUENESS, AND STABILITY of u_h, solution of (G)

To show the stability property, we start from coercivity of $a(\cdot,\cdot)$:

$$\exists\, \bar{\alpha} > 0 : \quad a(u_h, u_h) \geq \bar{\alpha} \, \| u_h \|_V^2$$

and we can do this because $u_h \in V_h \subset V$

Rearrange: $\quad \| u_h \|_V^2 \leq \dfrac{1}{\bar{\alpha}} \, a(u_h, u_h)$

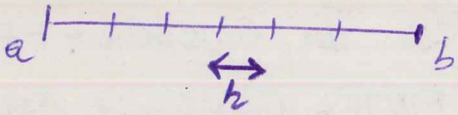
Use (G): $\quad \| u_h \|_V^2 \leq \dfrac{1}{\bar{\alpha}} \, F(u_h)$

use CONTINUITY $\equiv$ BOUNDEDNESS of F : $\quad \| u_h \|_V^2 \leq \dfrac{1}{\bar{\alpha}} \, C_F \| u_h \|_V$

$|F(v)| \leq C_F \| v \|_V \quad \forall v \in V$

$$\| u_h \|_V \leq \frac{C_F}{\bar{\alpha}} = \frac{\| F \|_{V'}}{\bar{\alpha}} \qquad \left(\| F \|_{V'} := \sup_{v \in V} \frac{|F(v)|}{\|v\|} \right)$$

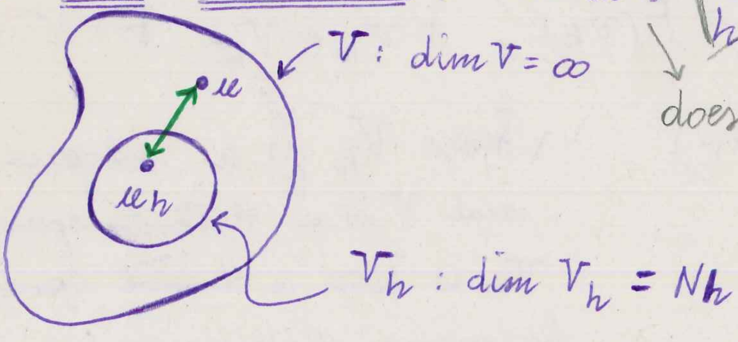
NICE RESULT: IT SHOWS THAT u_h IS STABLE INDEPENDENTLY of h (or N_h)



NOW OUR QUESTIONS ARE:

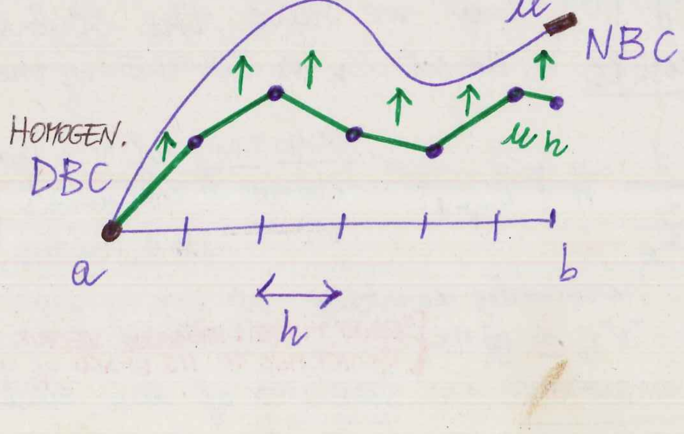
① IS u_h CONVERGENT? i.e. does $\cancel{\left(\Longleftrightarrow \lim_{N_h \to \infty} u_h = u \right)}$ $\lim_{h \to 0} u_h = u$? $\qquad N_h \sim \dfrac{1}{h}$

does $\lim_{h \to 0} \| u - u_h \|_V = 0$?



② and, if it is convergent, HOW does u_h converge to u ?

Qualitative picture



How the error $\| u - u_h \|_V$ decays as we refine the mesh, i.e. as $h \to 0$?

ORDER of CONVERGENCE IN h?

To answer ①, we first work out an important result

2 known as Céa lemma. that we need to prove the convergence of (G).

$$\left(\cancel{\text{"}\lim_{h\to 0} u_h = u\text{"}}\right) \iff \lim_{h\to 0} \|u-u_h\|_V = 0$$

"distance" btw the exact sol. & the approximate sol.

Let's look at $\|u-u_h\|_V$:

$$\|u-u_h\|_V^2 \le \frac{1}{\bar\alpha}\, a(u-u_h, u-u_h)$$

COERCIVITY of $a(\cdot,\cdot)$

Sum & Subtract $\forall w_h \in V_h$ in second argument of $a(\cdot,\cdot)$:

$$\|u-u_h\|_V^2 \le \frac{1}{\bar\alpha}\, a\big(u-u_h, (u-w_h)+(w_h-u_h)\big)$$

Use $a(\cdot,\cdot)$ - linearity in 2nd argument:

$$\|u-u_h\|_V^2 \le \frac{1}{\bar\alpha}\, a(u-u_h, u-w_h) + \overbrace{\frac{1}{\bar\alpha}\, a(u-u_h, \underbrace{w_h-u_h}_{=:v_h\in V_h})}^{\text{WE CLAIM } \emptyset \text{ (G.O.)}}$$

To prove this claim, we use

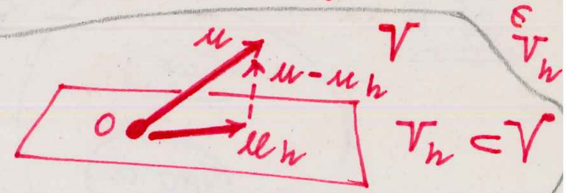
Galerkin Orthogonality (G.O.)

$$\left\{\begin{array}{l}(W) \text{ put } v=v_h \quad \forall v_h\in V_h \\ (G)\end{array}\right. \Rightarrow \left\{\begin{array}{l} a(u,v_h)=F(v_h) \\ a(u_h,v_h)=F(v_h)\end{array}\right. \xrightarrow{(W)-(G)} a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$$

USE $a(\cdot,\cdot)$ - LINEARITY IN THE 1st ARG.

CONFRONTA $a(\cdot,\cdot)$ CON LE PROPRIETÀ DEL PRODOTTO SCALARE $\Longleftarrow$

Qualitatively:



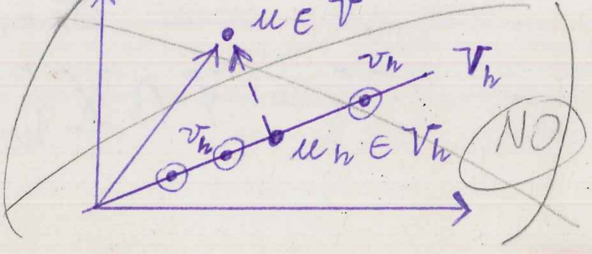
Use continuity of $a(\cdot,\cdot)$:

$\exists M>0: |a(u,v)| \le M\|u\|_V\|v\|_V \quad \forall v,u \in V$

$$\|u-u_h\|_V^{\cancel{2}} \le \frac{1}{\bar\alpha} M\, \cancel{\|u-u_h\|_V}\, \|u-w_h\|_V \qquad \forall w_h \in V_h$$

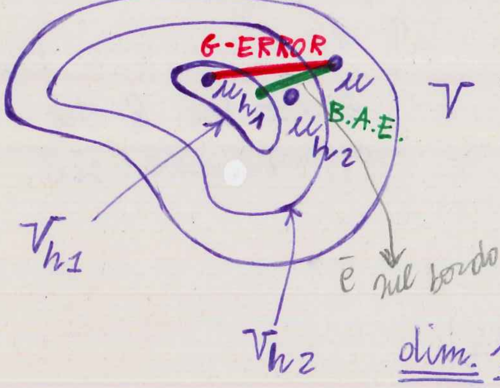
Now since the LHS is bounded by the RHS for ANY $w_h \in V_h$, we can take the Greatest Lower Bound of the RHS and write:

$$\underbrace{\boxed{\|u-u_h\|_V}}_{\text{GALERKIN ERROR}} \le \frac{M}{\bar\alpha} \underbrace{\boxed{\inf_{v_h\in V_h} \|u-v_h\|_V}}_{\text{B.A.E.}}$$



Céa lemma is an OPTIMALITY RESULT: the actual error (LHS) decays as the best approximation (RHS) that we can find on that subspace.

$\boxed{3}$ Can we improve the approximation? Yes, choose a larger subspace V_h



~~$N_h = \dim V_h$~~

$\Rightarrow$ IN PARTICULAR, WHAT WE DESIRE IS THAT:

~~" $\lim_{N_h \to \infty} V_h = V$ "~~ $\Rightarrow$ " $\lim_{h \to 0} V_h = V$ ",

in the sense that $\exists$ a family of finite dim. subspaces $\{V_h\}_{h>0} \mid V_h \subset V$ s.t.

contenuti

(APPROXIMABILITY CONDITION) $\quad \lim_{h \to 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0, \quad \forall v \in V.$

almost the B.A.E.

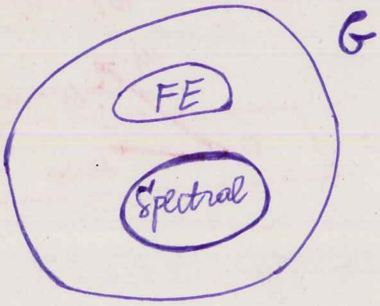
i.e. in the limit $h \to 0$, V_h becomes DENSE in V.

THIS CONDITION $\oplus$ CÉA LEMMA = CONVERGENCE of (G)

Proposition (Convergence of Galerkin)

Under the assumptions of Céa lemma (or Lax Milgram theorem) (V Hilbert, $a(\cdot,\cdot): V \times V \to \mathbb{R}$ bilinear, continuous & coercive, $F: V \to \mathbb{R}$ linear & continuous) and the additional assumption $\lim_{h \to 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0, \ \forall v \in V$ for a sequence of subspaces $\{V_h\}_{h>0} \subset V$, we have that (G) converges to (W), i.e. $\lim_{h \to 0} \|u - u_h\|_V = 0$.

N.B.:



- There are many Galerkin methods!
- The choice of basis for V_h determines the method.

e.g.1) Fourier basis $\longrightarrow$ Spectral Fourier method

e.g.2) Piecewise continuous polyn. of degree r X_h^r $\longrightarrow$ F.E.M.

GALERKIN - FINITE ELEMENT METHOD: we chose V_h such that

$$V_h = V \cap X_h^r \qquad X_h^r = \left\{ v_h \in \mathcal{C}^0([a,b]) : v_h|_{[x_{j-1}, x_j]} \in \mathbb{P}^r, \ j = 1, \dots, n \right\}$$

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We know that (G) converges, by previous proposition. But once we have chosen $V_h = V \cap X_h^r$, can we say something about HOW the G-FEM solution u_h converges to u?

IDEA: combine Céa lemma with results from approximation theory (of data & functions (i.e. Interpolation)). We will get an "a priori error estimation" of $\|u - u_h\|_V$

$\boxed{4}$

~~When the bound involves the exact solution.~~

def. interpolation operator: maps a continuous function into a piecewise continuous polynomial of degree r

$$\Pi_h^r : \mathcal{C}^0([a,b]) \to X_h^r$$

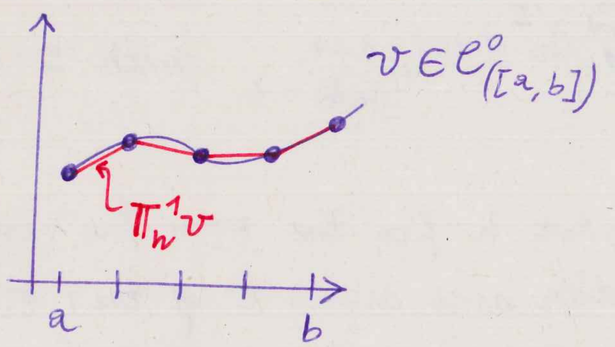
defined by $v \longmapsto \Pi_h^r v$

s.t. $\Pi_h^r v(\xi) = v(\xi)$ ~~(for $v \in \mathcal{C}^0([a,b])$)~~

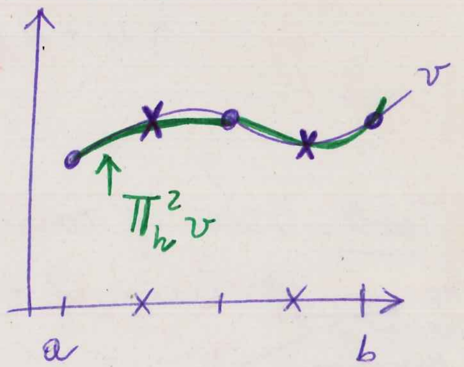
being ξ a node or vertex of the mesh.

example

1)



2)



From Céa lemma: $\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V \leq \frac{M}{\alpha} \|u - v_h\|_V \quad \forall v_h \in V_h$

We chose $v_h = \Pi_h^r u$: $\underbrace{\|u - u_h\|_V}_{\text{GALERKIN ERROR}} \leq \frac{M}{\alpha} \underbrace{\|u - \Pi_h^r u\|_V}_{\text{INTERPOLATION ERROR}}$

Use result from approximation theory...

Theor (Interp. error) [NO PROOF]

For $k = 0,1$ and $r \geq 1$, $\exists$ a constant $C_{k,r}$ s.t.

$$|v - \Pi_h^r v|_{H^k_{(a,b)}} \leq \underbrace{C_{k,r}}_{\uparrow} h^{r+1-k} |v|_{H^{r+1}_{(a,b)}} , \quad \forall v \in H^{r+1}_{(a,b)}$$

independent of h and v

NB. • if $k=0$ we get $\|\cdot\|_{L^2}$ on the LHS & h^{r+1} on the RHS
• if $k=1$ " " $|\ |_{H^1}$ " " " " h^r " " "

A PRIORI ERROR ESTIMATE (Th. 4.3 p. 75)

Prop. if $u \in H^{p+1}_{(\Omega)} \cap V$, $p \geq r \geq 1$ (r → degree of the interpolant), THEN $\exists$ C (indep. of h & u) s.t.

$$\|u - u_h\|_V \leq \frac{M}{\alpha} C h^r |u|_{H^{r+1}_{(a,b)}} , \quad u_h \in V_h \subset V$$

solution of G.FEM

IT IS CALLED A PRIORI BECAUSE THE BOUND INVOLVES THE EXACT SOLUTION!

What happens if $p \ngeq r$? A slightly more general result...

Prop if $u \in H^{p+1}_{(a,b)} \cap V$, $p > 0$, $r \geq 1$, THEN $\exists$ $\tilde{C}$ s.t.

$$\|u - u_h\|_V \leq \tilde{C} h^s |u|_{H^{s+1}_{(a,b)}} , \quad \text{with } s = \min\{p, r\}.$$

Table. Order of convergence w.r.t. h for the FEM for varying smoothness of the solution and degree r of the F.E.'s.

THE PROPOSITION IS NO MORE TRUE

r	$u \in H^1$, $\cancel{p=0 > 0}$	$u \in H^2$, $p=1$	$u \in H^3$, $p=2$	$u \in H^4$, $p=3$	$u \in H^5$, $p=4$
1	converges	$\boxed{h^1}$	h^1	h^1	h^1
2	"	h^1	$\boxed{h^2}$	h^2	h^2
3	"	h^1	h^2	$\boxed{h^3}$	h^3
4	"	h^1	h^2	h^3	$\boxed{h^4}$

due to the convg. of Galerkin probelm

Moral: increasing the degree of the F.E.'s is not enough to improve the convergence order, we also have to change space!!!

Esempio?

After the table:

Strategies to increase accuracy: ① Reducing h (refining the mesh)
② Using Finite Elements of higher degree.

The latter strategy makes sense only if the solution u is regular ("smooth") enough. (the regularity of the solution depends on the nature of the problem)

- From the Thm. it follows that the max value of r that it makes sense to take is $r = p$.

- Values $r > p$ do NOT ensure a better convergence.
If the solution is not very regular it is not convenient to use F.E.'s of high degree, because THE GREATER COMPUTATIONAL COST IS NOT BALANCED BY AN IMPROVEMENT of the CONVERGENCE.

Prop. If $u \in H^{p+1}_{(I)} \cap V$, $p > 0$, $r \geq 1$, $\exists\, \tilde{\tilde{C}}$ s.t.

$$\|u - u_h\|_{L^2(I)} \leq \tilde{\tilde{C}}\, h^{s+1}\, |u|_{H^s(I)},$$

$$s = \min\{p, r\}$$

(if we have smooth enough coefficients in the problem
& smooth enough domain (NO CORNERS...)
& right setting of BC's
THEN we will get the $+1$)